Power Flow Equation Formulation

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This document is a description for how to formulate the power flow equation. The current version is for Newton-Raphson AC power flow.

Basic Equations

We start with the basic equations

$$\mathbf{I} = \mathbf{Y} \cdot \mathbf{V} \tag{1}$$

$$I_i = \sum_{k=1}^{N} Y_{ik} V_k \tag{2}$$

$$\begin{aligned} S_i &= V_i I_i^* \\ &= V_i \left(\sum_{k=1}^{N} Y_{ik} V_k \right)^* \\ &= V_i \sum_{k=1}^{N} {Y_{ik}}^* {V_k}^* \end{aligned} \tag{3}$$

where N is the number of buses in the network, $\mathbf{V}$ is a vector of voltages, $\mathbf{I}$ is a vector of currents and $\mathbf{Y}$ is the Y matrix.

Suppose we define the complex voltages as

$$V_i = |V_i| e^{j\theta_i} \tag{4}$$

and then define the phase angle between bus i and bus k as

$$\theta_{ik} = \theta_i - \theta_k \tag{5}$$

Further, we can decompose the elements of the Y matrix into real and imaginary parts using

$$Y_{ik} = G_{ik} + jB_{ik} \tag{6}$$

The G_{ik} are called conductances and the B_{ik} are called susceptances.

Using these definitions we can write the expressions for the quantity **S** as

$$S_i = \sum_{k=1}^{N} |V_i||V_k|e^{j\theta_{ik}} \left(G_{ik} - jB_{ik}\right) \tag{7}$$

We can further decompose the S_i into real and imaginary parts using

$$S_i = P_i + jQ_i \tag{8}$$

This can be rewritten as

$$P_i + jQ_i = \sum_{k=1}^{N} |V_i||V_k| \left(\cos\theta_{ik} + j\sin\theta_{ik}\right)\left(G_{ik} - jB_{ik}\right) \tag{9}$$

Resolving equation (**??**) into real and imaginary parts gives the expressions

$$\begin{cases} P_i = \sum_{k=1}^{N} |V_i||V_k| \left(G_{ik}\cos\theta_{ik} + B_{ik}\sin\theta_{ik}\right) \\ Q_i = \sum_{k=1}^{N} |V_i||V_k| \left(G_{ik}\sin\theta_{ik} - B_{ik}\cos\theta_{ik}\right) \end{cases} \tag{10}$$

Newton-Raphson AC Power Flow

Now set up power flow equations in the form of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. The voltage magnitude and phase angle of the slack/reference bus (θ_1 and $|V_1|$) are known.

Define

$$\theta = \begin{bmatrix} \theta_2 \\ . \\ . \\ . \\ \theta_N \end{bmatrix} \quad (11)$$

$$|\mathbf{V}| = \begin{bmatrix} |V_2| \\ . \\ . \\ . \\ |V_N| \end{bmatrix} \quad (12)$$

$$\mathbf{x} = \begin{bmatrix} \boldsymbol{\theta} \\ |\mathbf{V}| \end{bmatrix} \quad (13)$$

The form of the vector function $\mathbf{f}(\mathbf{x})$ is

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} P_2(\mathbf{x}) - P_2 \\ . \\ . \\ . \\ P_N(\mathbf{x}) - P_N \\ ------- \\ Q_2(\mathbf{x}) - Q_2 \\ . \\ . \\ . \\ Q_N(\mathbf{x}) - Q_N \end{bmatrix} = - \begin{bmatrix} \Delta\mathbf{P}(\mathbf{x}) \\ \Delta\mathbf{Q}(\mathbf{x}) \end{bmatrix} = \mathbf{0} \quad (14)$$

where

$$\begin{cases} \Delta\mathbf{P} = P_i - P_i(\mathbf{x}) \\ \Delta\mathbf{Q} = Q_i - Q_i(\mathbf{x}) \end{cases} \quad (15)$$

P_i and Q_i can be obtained from the equation (**??**).

Now consider the Jacobian of $\mathbf{f}$, $\mathbf{J}$. The dimension of $\mathbf{J}$ using the expression for the complex voltage (**??**) is $(2(N-1))(2(N-1))$. The Jacobian itself has the form

$$\begin{aligned}\mathbf{J} &= \begin{bmatrix} \mathbf{J1} & \mathbf{J2} \\ \mathbf{J3} & \mathbf{J4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}} & \frac{\partial \mathbf{P}}{\partial |\mathbf{V}|} \\ \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\theta}} & \frac{\partial \mathbf{Q}}{\partial |\mathbf{V}|} \end{bmatrix}\end{aligned} \tag{16}$$

The N-R iteration itself has the form

$$\mathbf{J}\boldsymbol{\Delta}\mathbf{x} = -\mathbf{f}\left(\mathbf{x}\right) \tag{17}$$

Combining equations (**??**), (**??**) and (**??**) gives the expression

$$\begin{bmatrix} \mathbf{J1} & \mathbf{J2} \\ \mathbf{J3} & \mathbf{J4} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta}\theta \\ \boldsymbol{\Delta}|\mathbf{V}| \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Delta}\mathbf{P}\left(\mathbf{x}\right) \\ \boldsymbol{\Delta}\mathbf{Q}\left(\mathbf{x}\right) \end{bmatrix} \tag{18}$$

For the n^{th} iteration, we can rearrange equation (**??**) to get

$$\begin{bmatrix} \theta^{n+1} \\ |V|^{n+1} \end{bmatrix} = \begin{bmatrix} \theta^{n} \\ |V|^{n} \end{bmatrix} - \left[\mathbf{J}\left(x^n\right)\right]^{-1} \begin{bmatrix} \Delta P\left(x^n\right) \\ \Delta Q\left(x^n\right) \end{bmatrix} \tag{19}$$

The expressions for the elements of $\mathbf{J}$ can be found below:

$$J1 = \begin{cases} \frac{\partial P_i}{\partial \theta_i} = -\sum_{k\neq i} |V_i||V_k| \left(G_{ik}\sin\theta_{ik} - B_{ik}\cos\theta_{ik}\right) \\ \frac{\partial P_i}{\partial \theta_k} = |V_i||V_k| \left(G_{ik}\sin\theta_{ik} - B_{ik}\cos\theta_{ik}\right) \qquad i \neq k \end{cases} \tag{20}$$

$$J2 = \begin{cases} \frac{\partial P_i}{\partial |V_i|} = 2|V_i|G_{ii} + \sum_{k\neq i} |V_k| \left(G_{ik}\cos\theta_{ik} + B_{ik}\sin\theta_{ik}\right) \\ \frac{\partial P_i}{\partial |V_k|} = |V_i| \left(G_{ik}\cos\theta_{ik} + B_{ik}\sin\theta_{ik}\right) \qquad i \neq k \end{cases} \tag{21}$$

$$J3 = \begin{cases} \frac{\partial Q_i}{\partial \theta_i} = \sum_{k\neq i} |V_i||V_k| \left(G_{ik}\cos\theta_{ik} + B_{ik}\sin\theta_{ik}\right) \\ \frac{\partial Q_i}{\partial \theta_k} = -|V_i||V_k| \left(G_{ik}\cos\theta_{ik} + B_{ik}\sin\theta_{ik}\right) \qquad i \neq k \end{cases} \tag{22}$$

$$J4 = \begin{cases} \frac{\partial Q_i}{\partial |V_i|} = -2|V_i|B_{ii} + \sum_{k\neq i} |V_k| \left(G_{ik}\sin\theta_{ik} - B_{ik}\cos\theta_{ik}\right) \\ \frac{\partial Q_i}{\partial |V_k|} = |V_i| \left(G_{ik}\sin\theta_{ik} - B_{ik}\cos\theta_{ik}\right) \qquad i \neq k \end{cases} \tag{23}$$

The diagonal elements of $J1$ - $J4$ can also be written in the following form:

$$J1_{diag} = \frac{\partial P_i}{\partial \theta_i} = -Q_i - B_{ii}|V_i|^2 \tag{24}$$

$$J2_{diag} = \frac{\partial P_i}{\partial |V_i|} = P_i - G_{ii}|V_i|^2 \tag{25}$$

$$J3_{diag} = \frac{\partial Q_i}{\partial \theta_i} = \frac{P_i}{|V_i|} + G_{ii}|V_i| \tag{26}$$

$$J4_{diag} = \frac{\partial Q_i}{\partial |V_i|} = \frac{Q_i}{|V_i|} - B_{ii}|V_i| \tag{27}$$

where P_i and Q_i are given by equation (**??**).
The solution strategy is:

1. Assign initial voltage magnitudes and zero phase angles (θ) to all buses. For PQ buses(load buses), the voltage magnitudes are set equal to 1.0; for PV buses (voltage-controlled/generator buses), the voltage magnitudes are specified.
2. Calculate P_i, Q_i using equation (**??**).
3. Calculate ΔP_i, ΔQ_i using equation (**??**).
4. Calculate the element of Jacobian $\mathbf{J}$ using equation (**??**) to (**??**).
5. Solve the linear equation of (**??**) to find $\Delta\theta$ and $\Delta|V|$.
6. Update new voltage magnitudes and phase angles using equation (**??**).
7. Continue until the residuals of ΔP and ΔQ are less than the specified tolerance.